

# LIFSHITZ TAILS AND LOCALIZATION IN 3D ANDERSON MODEL

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**ABSTRACT.** Consider the 3D Anderson model with a zero mean and bounded i.i.d. random potential. Let  $\lambda$  be the coupling constant measuring the strength of the disorder, and  $\sigma(E)$  the self energy of the model at energy  $E$ . For any  $\epsilon > 0$  and sufficiently small  $\lambda$ , we derive almost sure localization in the band  $E \leq -\sigma(0) - \lambda^{4-\epsilon}$ . In this energy region, we show that the typical correlation length  $\xi_E$  behaves roughly as  $O(|E| - \sigma(E))^{-1/2}$ , completing the argument, outlined in the unpublished work of T. Spencer [18].

## 1. INTRODUCTION, MAIN RESULT AND STEPS OF THE PROOF

**1.1. Introduction.** In this paper we want to carry out the program, sketched in the unpublished preprint of T. Spencer [18], regarding the localization for the 3D Anderson model in the so-called Lifshitz tails regime.

The *Anderson* operator  $H_\omega^\lambda$  on the lattice  $\mathbb{Z}^3$  acts on the vector  $\psi \in \ell^2(\mathbb{Z}^3)$  as:

$$(H_\omega^\lambda \psi)(n) := -\frac{1}{2}(\Delta \psi)(n) + \lambda V_\omega(n)\psi(n), \quad (1.1)$$

where  $\Delta$  denotes the discrete Laplace operator,

$$(\Delta \psi)(n) = \sum_{e \in \mathbb{Z}^3, |e|=1} \psi(n+e) - 6\psi(n).$$

We will assume throughout this paper

A1 The values of the random potential  $V_\omega(\cdot)$  are independent, identically distributed variables, with even, compactly supported, and bounded probability density  $\rho$ .

A2 For any  $m \in \mathbb{N}$ ,  $\mathbb{E}(V_\omega^{2m}(x)) \leq c$  with some constant  $c$ , and  $\mathbb{E}(V_\omega^2(x)) = 1$ .

Let  $e(p)$  denote the dispersion law, associated with the Fourier transform of the Laplacian,  $(\mathcal{F}\Delta f)(p) = -2e(p)\hat{f}(p)$ , where

$$\hat{f}(p) := (\mathcal{F}f)(p) := \sum_{n \in \mathbb{Z}^3} e^{-i2\pi p \cdot n} f(n), \quad p \in \mathbb{T}^3 := [-1/2, 1/2]^3,$$

with its inverse

$$\check{g}(n) = \int_{\mathbb{T}^3} d^3p e^{i2\pi p \cdot n} f(p).$$

One then computes

$$e(p) = 2 \sum_{\alpha=1}^3 \sin^2(\pi p \cdot e_\alpha), \quad (1.2)$$

where  $e_\alpha$  is a unit vector in the  $\alpha$  direction. The spectrum of the unperturbed operator  $H_\omega^0$  is absolutely continuous and consists of the interval  $[0, 6]$ .

In what follows we will denote by  $A(x, y)$  the kernel of the linear operator  $A$  acting on  $l^2(\mathbb{Z}^3)$  (that is  $A(x, y) = (\delta_y, A\delta_x)$ , where  $\delta_x$  is an indicator function of the site  $x \in \mathbb{Z}^3$ , and  $(\cdot, \cdot)$  denotes the inner product of  $l^2(\mathbb{Z}^3)$ ). We will use the concise notation  $\int$  in place of  $\int_{(\mathbb{T}^3)^k}$  whenever it is clear from the context that each of the  $k$  variables of integration is integrated out over a torus  $\mathbb{T}^3$ .

We will investigate the properties of  $H_\omega^\lambda$  for a typical configuration  $\omega$  in a weak disorder regime, namely at the energy range

$$\left[ \lambda a, -\lambda^2 \int_{\mathbb{T}^3} \frac{d^3 p}{e(p)} - \lambda^{4-\epsilon} \right],$$

with  $a = \min\{x : x \in \text{support } \rho\}$  for any  $\epsilon > 0$  and  $\lambda > 0$  being sufficiently small<sup>1</sup>. Most of the mathematical results on localization for operators with random potential in dimensions  $d > 1$  have been derived using the multi-scale analysis introduced by Fröhlich and Spencer [8] and by the fractional moment method of Aizenman and Molchanov [2] (we are going to use the latter approach). By now there exists extensive general literature on the Anderson localization problem, see for example [19] and references therein.

The quantity of the most interest is the typical asymptotic behavior of the so called Green function (also known as the two point correlation function, the propagator)

$$R(x, y) = (H_\omega^\lambda + E + i\eta)^{-1}(x, y)$$

in the limit  $\eta \searrow 0$ . It plays a crucial role in determining, for instance, the conductivity properties of the physical sample (whether it is an insulator or a conductor at a given energy band). On a mathematical level, investigation of the propagator can yield an insight on the typical spectrum of  $H_\omega^\lambda$  at the vicinity of  $-E$ . The Anderson model (1.1) is characterized by the following dichotomy, [3]: Either the typical Green function  $R(x, y)$  decays at least exponentially fast when  $|x - y| \rightarrow \infty$ , or it cannot decay faster than  $|x - y|^{-6}$  in a three dimensional case. The former behavior is called *localization* and necessitate that the spectrum is pure point almost surely in the vicinity of  $-E$ , [2]. The localization region is naturally characterized by the so called *correlation length*  $\xi_E$ , that is a typical length scale  $|x - y|$  at which  $R(x, y)$  starts to decay at least exponentially fast. We stress here the energy dependance of the correlation length, for it is going to play a role in our analysis. The consensus among the condensed matter physicists is that in  $3D$ , in the weak disorder regime, there should be a spectral transition from point spectrum to continuous one. This phenomenon is known as the Anderson transition. The proof of such an actuality presents a great challenge in this subject. The region of the possible spectral transition is called the *mobility edge*. One can get a certain indication of the existence of the delocalized regime for the Anderson model by showing that there occurs a threshold energy  $E_0$  (which presumably coincides with the mobility edge of the problem) such that  $\xi_E$  diverges as  $E$  approaches  $E_0$ . The present work can be seen as a step in this direction.

The occurrence of localization at energies near the band edges at weak disorder is related to the rarefaction of low eigenvalues, and was already discussed in the physical literature by I. M. Lifshitz in 1964, see Section 3 in [13], and [14]. As far as the rigorous results are concerned, let us only mention the three closely related works: M. Aizenman [1] showed that the spectrum of  $H_\omega^\lambda$  consists (almost surely)

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<sup>1</sup> $\lambda a = \inf(E : E \in \sigma(H_\omega^\lambda))$  almost surely, see e.g. [19] for details.

of the localized eigenvalues at the energy range  $[\lambda a, \lambda a + \lambda^\alpha]$ , with  $\alpha = 5/4$ . This result was later improved by W-M. Wang [20] ( $\alpha = 1$ ), whose result was in turn enhanced by F. Klopp [10], who extended the region of localization all the way up to the (negative) energies of order  $\lambda^{1+1/6}$  in 3D. In this work we push the upper bound of the localization region further up, to the value  $-C\lambda^2$ , and examine the behavior of the correlation length as a function of energy.

**1.2. Results.** In order to formulate our main technical accomplishment we need to introduce some further notation. The self energy term  $\sigma(E)$ , associated with  $H_\omega^\lambda$ , is given by the solution of the self-consistent equation<sup>2</sup>

$$\sigma(E) = \lambda^2 \int_{\mathbb{T}^3} \frac{d^3 p}{e(p) + E - \sigma(E)}. \quad (1.3)$$

It is easy to check that  $\sigma(E)$  is positive and uniformly bounded by  $C\lambda^2$  with some constant  $C$ , provided that

$$E > E_0 := \lambda^2 \int \frac{d^3 p}{e(p)}$$

and  $E^* := E - \sigma(E) > 0$  for such values of  $E$  (the relevant properties of the solution of (1.3) are collected in Appendix A. Moreover, if

$$E \geq E_\epsilon(\lambda) := \lambda^2 \int_{\mathbb{T}^3} \frac{d^3 p}{e(p)} + \lambda^{4-\epsilon}, \quad (1.4)$$

then  $E^* > C\lambda^{4-\epsilon}$  for an arbitrary small  $\epsilon$  and sufficiently small values of  $\lambda$ . We therefore can define a (renormalized) free Green function

$$R_r(x, y) = (-\frac{1}{2}\Delta + E - \sigma(E) + i0)^{-1}(x, y) \quad (1.5)$$

for every  $E$  in the above energy range. Let us also denote by

$$R(x, y) = (H_\omega^\lambda + E + i0)^{-1}(x, y), \quad (1.6)$$

which is well defined a.s., [3].

The hallmark of localization is rapid decay of  $G(x, y)$  at energies in the spectrum of  $H_\omega$ , for the typical configuration  $\omega$ . Rapid decay of the Green function is related to the non-spreading of wave packets supported in the corresponding energy regimes and various other manifestations of localization whose physical implications have been extensively studied in regards to the conductive properties of metals and in particular to the quantum Hall effect. Our main result (Theorems 1 and 2 below) establishes this behavior of the Green function at the band edges of the spectrum, by comparing it with the asymptotics of the free Green function  $R_r(x, y)$ . The behavior of the latter for  $|x - y| \gg 1$  is known [9]:

$$R_r(x, y) \sim \frac{1}{2\pi|x - y|} e^{-\sqrt{2E^*}|x - y|}.$$

The implication is that the correlation length for the free Green function is  $(E^*)^{-1/2}$ . Moreover, for the energy range (1.4) we have

$$(|x - y| + 1)^{-1} \gg \lambda(|x - y| + 1)^{-1/2}$$

whenever

$$|x - y| < (E^*)^{-1/2},$$

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<sup>2</sup>Note that since we will be interested at negative energies,  $E$  in (1.3) is assumed to be positive.

hence

$$|R_r(x, y)|^s \gg \frac{\lambda^s}{(|x - y| + 1)^{s/2}}, \quad \text{for } |x - y| < (E^*)^{-1/2}, \quad E \geq E_\epsilon(\lambda).$$

With this estimate in mind, we present

**Theorem 1** (Local fractional moment bound). *For  $H_\omega^\lambda$  as above, for any  $s < 1/2$  and  $\epsilon > 0$ , there exists  $\lambda_0(\epsilon)$  such that for all  $\lambda < \lambda_0(\epsilon)$  and  $E \geq E_\epsilon(\lambda)$ , one has a bound*

$$\mathbb{E} |R(x, y) - R_r(x, y)|^s \leq C_1(s) \frac{\lambda^s}{(|x - y| + 1)^{s/2}}, \quad (1.7)$$

with  $C_1(s) < \infty$ , which holds for any pair  $\{(x, y) \in \mathbb{Z}^3 \times \mathbb{Z}^3 : |x - y| < (E^*)^{-1/2}\}$ .

The estimate (1.7) can be interpreted as follows: The typical correlation length  $\xi_E$  within this energy range cannot be smaller than  $O((E^*)^{-1/2})$ , in particular it grows as  $E^*$  approaches zero<sup>3</sup>. The next proposition shows that  $\xi_E$  cannot be of much greater scale either.

**Theorem 2** (Global fractional moment bound). *For  $H_\omega^\lambda$  as above, there exists  $\lambda_0(\epsilon)$ , so that for all  $\lambda < \lambda_0(\epsilon)$ ,  $s < 1/4$ , and  $E \geq E_\epsilon(\lambda)$ , one has a bound*

$$\mathbb{E} |R(x, y)|^s \leq \frac{C_2(s)}{\lambda^s} e^{C_3 \sqrt{E^*} \ln(E^*) |x - y|} \quad (1.8)$$

for all  $x, y \in \mathbb{Z}^3$ .

Let us list several known implications of the global fractional moment bound:

- i. *Spectral localization* ([2]): The spectrum of  $H_\omega$  within the interval (1.4) is almost-surely of the pure-point type, and the corresponding eigenfunctions are exponentially localized.
- ii. *Dynamical localization* ([1]): Wave packets with energies in the specified range do not spread (and in particular the *SULE* condition of [16] is met):

$$\mathbb{E} \left( \sup_{t \in \mathbb{R}} | (P_{\{H_\omega^\lambda < -E_\epsilon\}} e^{-itH}) (x, y) | \right) \leq \tilde{A} e^{-\tilde{\mu} |x - y|}, \quad (1.9)$$

where  $P_{H < a}$  stands for the spectral projection of  $H$  on the energies below  $a$ .

- iii. *Absence of level repulsion* ([15]). Minami has shown that (1.8) implies that in the range (1.4) the energy gaps have Poisson-type statistics.

For energies  $E$  slightly above  $-E_0$  it is expected on the physical grounds (see also a discussion below) that  $H_\omega^\lambda$  should almost surely have absolutely continuous spectrum in  $3D$ . Presumably, the correlation length truly diverges when one starts to approach the mobility edge and it will be extremely interesting to cover the missing case of  $E \in [-E_\epsilon(\lambda), -E_0]$ .

This result can also be established by studying the density of states. Once the DOS is shown to be small below  $E^*$ , the localization is a rather straightforward consequence of known methods (as say in [10]). However, if one wants to study the behavior of the model at the closer vicinity of  $E^*$ , this extra step can be an obstacle, as DOS increases.

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<sup>3</sup>Unfortunately, in this work we are only able to descend to the values  $E^* = O(\lambda^{4-\epsilon})$ , so we cannot claim that the correlation length indeed diverges.

As was pointed to us by the referee, the numerical results seem to qualitatively agree with the suggestion that the mobility edge is near  $-C\lambda^2$  (e.g. [17]). If one uses Cauchy variables instead of the box distribution, it is possible to calculate certain quantities explicitly for such variables:  $\mathbb{E}(f(v)) = f(i)$  for a function  $f$  having a bounded analytic continuation to the upper half plane. In particular, DOS is computable, and it is not small for negative energies (namely  $O(\sqrt{\lambda})$ ). The lower mobility edge in  $d = 3$  appears to be positive on the basis of numerical studies (e.g. [11], Figure 5), which suggests that the existence of high moments of the distribution of the potential plays a crucial role in the analysis.

**1.3. Major steps in the proof.** In the unpublished notes, T. Spencer [18] proposed to prove the localization near the band edge using the multiscale analysis, with the initial volume estimates coming from the fact that the density of states in the Lifshitz tail regime is small. To control the density of states he suggested to truncate the resolvent expansion at some optimal point. The corresponding Feynman graphs become superficially convergent after the suitable renormalization. In this paper, we complete the proof of the result announced in [18], combining Spencer's perturbative approach with the Aizenman-Molchanov fractional moment method and developing the detailed estimates on the error terms in the renormalized expansion.

The following representation for a Green function  $R(x, y)$  will be useful:

**Lemma 1.1.** *For any integer  $N$  and energies  $E$  that satisfy (1.4) we have the decomposition*

$$R(x, y) = \sum_{n=0}^{N-1} A_n(x, y) + \sum_{z \in \mathbb{Z}^3} \tilde{A}_N(x, z) R(z, y), \quad (1.10)$$

with  $A_0(x, y) = R_r(x, y)$ , and where the (real valued) kernels  $A_n, \tilde{A}_N$  satisfy bounds

$$\mathbb{E}(A_n(x, y))^2 \leq (4n)! E^* \left( C(E^*) \frac{\lambda^2}{\sqrt{E^*}} \right)^n e^{-\sqrt{\frac{E^*}{3}} |x-y|}, \quad n > 1; \quad (1.11)$$

$$\mathbb{E}|\tilde{A}_N(x, y)| \leq \sqrt{(4N)!} \left( C(E^*) \frac{\lambda^2}{\sqrt{E^*}} \right)^{N/2} e^{-\sqrt{\frac{E^*}{12}} |x-y|}, \quad N > 1; \quad (1.12)$$

where  $C(E^*) = K \ln^9 E^*$  for some generic constant  $K$ .

The zero order contribution  $A_0$  satisfies

$$0 < A_0(x, y) = \int_{\mathbb{T}^3} e^{i(x-y)p} \frac{d^3 p}{e(p) + E^*} \leq \frac{K}{(|x-y| + 1)} \quad (1.13)$$

for all  $x, y \in \mathbb{Z}^3$ , and behaves asymptotically as

$$A_0(x, y) = \left( 1 + O(\sqrt{E^*}) + O(|x-y|^{-1}) \right) \times \frac{e^{-\sqrt{2E^*}|x-y|}}{2\pi(|x-y| + 1)}. \quad (1.14)$$

Lastly, we have

$$\mathbb{E}(A_1(x, y))^2 \leq \frac{K \lambda^2}{|x-y| + 1} e^{-2\sqrt{2E^*}|x-y|}. \quad (1.15)$$

One then looks for the optimal value  $N$  to stop the expansion - note that the increasing factor of  $(4N)!$  in  $A_N(x, y)$  competes with the decreasing factor  $(\lambda^4 E^*)^{N/2}$ .

The choice  $\mathbb{E}^* > \lambda^{4-\epsilon}$  has the effect that

$$C(E^*) \frac{\lambda^2}{\sqrt{E^*}} \leq \lambda^{B\epsilon}, \quad 0 < B < 1, \quad (1.16)$$

which suffices to control (1.11) – (1.12). It turns out that the appropriate choice for  $N$  should satisfy

$$(4N)! \left( \frac{C(E^*)\lambda^2}{\sqrt{E^*}} \right)^N \approx e^{-N}$$

(see the next section for details). In terms of the  $\lambda$  - dependence, it corresponds to  $N \sim \lambda^{-b\epsilon}$  for  $b < B$ . Note that the square root in the denominator of (1.16) is absolutely crucial for the strategy. To this end, let us mention that the representation (1.10) is a resolvent type expansion (cf. Lemma 3.1 below). If one applies the rough norm bound on each factor of the resolvent there, the denominator in (1.16) will contain  $E^*$  rather than its square root. The improvement is achieved using the Feynman diagrammatic technique (Section 4).

Let us denote by  $H_\omega^{\Lambda,\lambda}$  the natural restriction of  $H_\omega^\lambda$  to  $\Lambda \subseteq \mathbb{Z}^3$  and let  $R^\Lambda = (H_\omega^{\Lambda,\lambda} + E + i0)^{-1}$  be the corresponding resolvent.

Theorems 1 and 2 follow from the result above and from Aizenman–Molchanov a-priori bound on the fractional moment of the Green function as it appears in Lemma 2.1 of [3], which states that

$$\mathbb{E} |R^\Lambda(x, y)|^s < C_s \quad (1.17)$$

for any  $0 < s < 1$ , uniformly in  $x, y \in \Lambda$  and  $\lambda$ . Moreover, the bound above holds uniformly for an arbitrary set  $\Lambda$ .

The rest of the paper is organized as follows: To make the presentation less obscure, we postpone the rather lengthy proof of the main technical Lemma 1.1 until Section 4 and establish first Theorems 1 and 2. In Section 3 we perform a self energy renormalization required to get rid of the so called tadpole contributions. A technical statement regarding the properties of the self energy term  $\sigma(E)$  is proven in Appendix A.

## 2. PROOFS OF THEOREM 1 AND 2

**2.1. Theorem 1.** Lemma 1.1, Aizenman–Molchanov a-priori bound (1.17), and Hölder inequality imply that for any  $0 < s < 1/2$

$$\begin{aligned}
& \mathbb{E} |R(x, y) - R_r(x, y)|^s \\
& \leq \sum_{l=1}^{N-1} \mathbb{E} A_l^s(x, y) + \sum_{z \in Z^3} \left( \mathbb{E} |\tilde{A}_N(x, z)|^{2s} \right)^{1/2} (\mathbb{E} |R(z, y)|^{2s})^{1/2} \\
& \leq \sum_{l=1}^{N-1} (\mathbb{E} A_l^2(x, y))^{s/2} + \sum_{z \in Z^3} \left( \mathbb{E} |\tilde{A}_N(x, z)|^2 \right)^s (\mathbb{E} |R(z, y)|^{2s})^{1/2} \\
& \leq \frac{K^{s/2} \lambda^s}{|x - y|^{s/2} + 1} e^{-s\sqrt{2E^*}|x-y|} \\
& \quad + \sum_{l=2}^{N-1} ((4l)!)^{s/2} (E^*)^{s/2} \left( \frac{C(E^*)\lambda^2}{\sqrt{E^*}} \right)^{sl/2} e^{-s\sqrt{\frac{E^*}{12}}|x-y|} \\
& \quad + C(s)((4N)!)^{s/2} \left( \frac{C(E^*)\lambda^2}{\sqrt{E^*}} \right)^{sN/2} \sum_{z \in Z^3} e^{-s\sqrt{\frac{E^*}{12}}|x-z|} \\
& \leq \frac{K^{s/2} \lambda^s}{|x - y|^{s/2} + 1} e^{-s\sqrt{2E^*}|x-y|} \\
& \quad + (C(E^*)\lambda^2)^s e^{-s\sqrt{\frac{E^*}{12}}|x-y|} \sum_{l=2}^{N-1} ((4l)!)^{s/2} \left( \frac{C(E^*)\lambda^2}{\sqrt{E^*}} \right)^{s(l-2)/2} \\
& \quad + \frac{\tilde{C}(s)}{(E^*)^{3/2}} ((4N)!)^{s/2} \left( \frac{C(E^*)\lambda^2}{\sqrt{E^*}} \right)^{sN/2}. \quad (2.1)
\end{aligned}$$

Choosing

$$(4N)^4 = \frac{\sqrt{E^*}}{C(E^*)\lambda^2}$$

one obtains, using the Stirling's approximation, that the summation over the index  $l$  is bounded by some  $s$ -dependent constant. On the other hand, for such  $N$  we have  $(4N)! \left( \frac{C(E^*)\lambda^2}{\sqrt{E^*}} \right)^N < e^{-N}$ . Hence, for such a value of  $N$  we have

$$\begin{aligned}
& \mathbb{E} |R(x, y) - R_r(x, y)|^s < C_s \left( \frac{\lambda^s e^{-s\sqrt{2E^*}|x-y|}}{|x - y|^{s/2} + 1} \right. \\
& \quad \left. + (C(E^*)\lambda^2)^s e^{-s\sqrt{\frac{E^*}{12}}|x-y|} + (E^*)^{-3/2} \exp \left( -\frac{s \sqrt[4]{\frac{\sqrt{E^*}}{C(E^*)\lambda^2}}}{8} \right) \right) \quad (2.2)
\end{aligned}$$

with some generic constant  $C_s$ . We infer that for any  $\epsilon > 0$  and sufficiently small  $\lambda_0(\epsilon)$  we have for any  $\lambda < \lambda_0(\epsilon)$  and  $E^* > \lambda^{4-\epsilon}$

$$\mathbb{E} |R(x, y) - R_r(x, y)|^s \leq C_s \frac{\lambda^s}{|x - y|^{s/2} + 1}, \quad |x - y| < (E^*)^{-1/2}, \quad (2.3)$$

(that is we proved Theorem 1) and

$$\mathbb{E} |R(x, y)|^s \leq C'_s \left( \lambda^s e^{-s \frac{\sqrt{E^*}}{4} |x-y|} + e^{-s \lambda^{-\epsilon/9}} \right), \quad (2.4)$$

for any  $x, y \in \mathbb{Z}^3$ , where we have used the bound (1.14) for  $R_r(x, y)$ . The latter estimate will be used in the proof of Theorem 2.

**2.2. Theorem 2.** Using notation introduced after (1.17) we define the decoupled Hamiltonian  $H_\Lambda$  to be:

$$H_\Lambda = H_\omega^{\Lambda, \lambda} \oplus H_\omega^{\Lambda^c, \lambda},$$

where  $\Lambda$  is a cubic box of the linear size  $2L$  centered around the origin, and  $\Lambda^c := \mathbb{Z}^3 \setminus \Lambda$ . Let  $\partial\Lambda$  be a boundary  $\Lambda$ . We will denote  $R^\Lambda := (H_\Lambda + E^*)^{-1}$ . For any  $s < 1/2$  and  $n \in \partial\Lambda$  we have

$$\begin{aligned} \mathbb{E} |R^\Lambda(n, 0)|^s &\leq \mathbb{E} |R(n, 0)|^s + \mathbb{E} |R^\Lambda(n, 0) - R(n, 0)|^s \\ &= \mathbb{E} |R(n, 0)|^s + \mathbb{E} |(R^\Lambda (H_\Lambda - H_\omega^\lambda) R)(n, 0)|^s \\ &\leq \mathbb{E} |R(n, 0)|^s \\ &\quad + \sum_{\text{dist}(k, \partial\Lambda) \leq 1} \left\{ \mathbb{E} |(R^\Lambda (H_\Lambda - H_\omega^\lambda))(n, k)|^{2s} \right\}^{1/2} \left\{ \mathbb{E} |R(k, 0)|^{2s} \right\}^{1/2}, \end{aligned} \quad (2.5)$$

where we used locality of  $(H_\Lambda - H_\omega^\lambda)$  - its non-zero matrix elements lies essentially on the boundary of  $\Lambda$ . Similar considerations lead to the estimate

$$\mathbb{E} |(R^\Lambda (H_\Lambda - H_\omega^\lambda))(n, k)|^{2s} \leq C \sup_{m: \text{dist}(m, \partial\Lambda) \leq 1} \mathbb{E} |R^\Lambda(n, m)|^{2s}. \quad (2.6)$$

Using bounds (2.6) and (1.17) as well as the Hölder inequality, we obtain from (2.5)

$$\mathbb{E} |R^\Lambda(n, 0)|^s \leq C_s \sum_{k \in \partial\Lambda} \left\{ \mathbb{E} |R(k, 0)|^{2s} \right\}^{1/2}. \quad (2.7)$$

Plugging the bound (2.4) into the latter equation, with  $s < 1/4$ , we establish

$$\mathbb{E} |R^\Lambda(n, 0)|^s \leq C_s L^2 \left( \lambda^s e^{-s L \sqrt{\frac{E^*}{12}}} + e^{-s \lambda^{-\epsilon/9}} \right) \quad (2.8)$$

Now we are in a position to use the fractional moment criterion [3], Theorem 1.2, which states that if

$$B_s L^4 \lambda^{-2s} \sum_{n \in \partial\Lambda} \mathbb{E} |R^\Lambda(n, 0)|^s < b, \quad (2.9)$$

for a certain constant  $B_s$  and  $b < 1$ , then we have a bound

$$\mathbb{E} |R(x, y)|^s \leq \frac{B_s}{b^2 \lambda^s} e^{\frac{\ln b}{L} |x-y|}. \quad (2.10)$$

It is easy to see from (2.8) that (2.9) is satisfied, provided that  $E^* > \lambda^{4-\epsilon}$ , and  $\lambda$  is small enough<sup>4</sup>, with

$$L = O \left( \frac{\ln(E^*)^{-1}}{s \sqrt{E^*}} \right),$$

hence the result.

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<sup>4</sup>Since the bottom of the spectrum of  $H_\omega$  is almost surely located at  $\lambda a$ , we can assume without loss of generality that  $E^* \leq |a|\lambda$ .



## 3. RENORMALIZATION OF TADPOLE'S CONTRIBUTION

**3.1. Expectation of the product of the random potentials.** In what follows, the expectation of the product of random variable  $V_\omega$ , namely

$$\mathbb{E} \left[ \prod_{j=1}^n V_\omega(x_j) \right]$$

will play an important role. These products naturally arise when one starts to expand the operator  $R := (H_\omega^\lambda + E + i0)^{-1}$  in the resolvent series

$$R = \sum_{i=0}^n (-\lambda R_0 V_\omega)^i R_0 + (-\lambda R_0 V_\omega)^{n+1} R$$

about the (unperturbed) operator  $R_0 := (H_0 + E)^{-1}$ . Indeed,

$$(-\lambda R_0 V_\omega)^n R_0(x_0, x_{n+1}) = (-\lambda)^n \sum_{x_j \in \mathbb{Z}^3; j=1, \dots, n} \prod_{j=1}^n V_\omega(x_j) \prod_{i=0}^n R_0(x_i, x_{i+1}). \quad (3.1)$$

Let  $\Upsilon_{N,N'}$  be the set  $\{1, \dots, N, N+2, \dots, N+N'+1\}$ , while  $\Pi_{N,N'}$  will denote the set of partitions of  $\Upsilon_{N,N'}$  into disjoint subsets  $S_j$  of size  $|S_j| \in 2\mathbb{N}$ . Two partitions  $\pi = \{S_j\}_{j=1}^m$ ,  $\pi' = \{S'_j\}_{j=1}^m$  are equivalent,  $\pi = \pi'$ , if they coincide up to the permutation. For  $S \subset \Upsilon_{N,N'}$ , let

$$\delta(x_S) = \sum_{y \in \mathbb{Z}^3} \prod_{j \in S} \delta_{|x_j - y|}, \quad (3.2)$$

where  $\delta_x$ ,  $x \in \mathbb{Z}$  is Kronecker delta function, and  $x_S$  denotes the collection of  $\{x_i, i \in S\}$ . One has an identity (see e.g. [5] Section 3.1 for details)

$$\mathbb{E} \left[ \prod_{j \in \Upsilon_{N,N}} V_\omega(x_j) \right] = \sum_{m=1}^N \sum_{\pi = \{S_j\}_{j=1}^m} \prod_{j=1}^m c_{|S_j|} \delta(x_{S_j}), \quad (3.3)$$

with coefficients  $c_{2l} \leq (cl)^{2l+1}$  proviso (A1-A2), and  $c_2 = \mathbb{E}(V_\omega^2(x)) = 1$ . The set  $S_j$  in the partitions  $\pi \in \Pi_{N,N'}$  can be of the special type: If

$$S_j = \{i, i+1\} \quad (3.4)$$

we will refer to it as a *tadpole*, or a *gate* set. The non zero order contributions in the resolvent expansion, associated with the tadpole-free terms, are sufficiently small in the energy range (1.4), as opposed to the contributions containing the gates.

**3.2. Self energy renormalization.** The purpose of renormalization is then to include the tadpole contributions into the propagator itself. In our case this can be established by subtracting from the unperturbed operator  $-\frac{1}{2}\Delta$  the self-energy term  $\sigma(E)$ , described in Section 1. We decompose  $H_\omega^\lambda$  as

$$H_\omega^\lambda = H_r + \tilde{V}, \quad H_r := -\frac{1}{2}\Delta - \sigma(E), \quad \tilde{V} := \lambda V_\omega + \sigma(E).$$

The corresponding resolvent expansion for  $R$  defined in (1.6) takes the form

$$R = \sum_{i=0}^n (-R_r \tilde{V})^i R_r + (-R_r \tilde{V})^{n+1} R, \quad (3.5)$$

with  $R_r$  as in (1.5). Note that for any  $x \in \mathbb{Z}^3$

$$\sigma(E) = \lambda^2 R_r(x, x). \quad (3.6)$$

In place of (3.1) we get

$$(-\lambda R_r \tilde{V})^n R_r(x_0, x_{n+1}) = \sum_{x_j \in \mathbb{Z}^3; j=1, \dots, n} \prod_{j=1}^n (-\lambda V_\omega(x_j) - \sigma(E)) \prod_{i=0}^n R_r(x_i, x_{i+1}) \quad (3.7)$$

If we open the brackets in (3.7), we obtain

$$\sum_{\theta, x_j \in \mathbb{Z}^3; j=1, \dots, n} R_r(x_0, x_1) \theta(x_1) R_r(x_1, x_2) \theta(x_2) \dots R_r(x_{n-1}, x_n) \theta(x_n) R_r(x_n, x_{n+1})$$

where  $\theta(x)$  is either  $-\lambda V_\omega(x)$ , or  $-\sigma(E)$  (whenever  $\theta(x) = -\sigma(E)$  we will refer to it as a *bullet*). Since  $\sigma(E) = O(\lambda^2)$  for all permissible values of  $E$ , see Appendix A, one can unambiguously define the *order*  $l$  (in powers of  $\lambda$ ) of the particular contribution

$$R_r(x_0, x_1) \theta(x_1) R_r(x_1, x_2) \theta(x_2) \dots R_r(x_{n-1}, x_n) \theta(x_n) R_\#(x_n, x_{n+1}),$$

(with  $R_\#$  being either  $R_r$  or  $R$ ) according to the following rule: Each factor of  $\sigma(E)$  counts as 2, while appearance of the random potential counts as 1, and we add up all the exponents to get the order of the term. For instance, the order of the expression

$$R_r(x_0, x_1) \sigma(E) R_r(x_1, x_2) \lambda V_\omega(x_2) R_r(x_2, x_3) \sigma(E) R(x_3, x_4)$$

is 5.

To handle the renormalization of tadpole contributions properly, we decide at which value of  $n$  to halt the expansion in (3.5) individually for each contribution according to the following rule (to which we will refer as a stopping rule): If the order of the kernel

$$R_r(x_0, x_1) \theta(x_1) R_r(x_1, x_2) \theta(x_2) \dots R_r(x_{n-1}, x_n) \theta(x_n) R(x_n, x_{n+1}) \quad (3.8)$$

reaches or exceeds a value  $N$  to be determined later on, we stop expanding this particular term. To illustrate this procedure we write down the expansion obtained in a case of  $N = 2$ :

$$\begin{aligned} R &= R_r - R_r \sigma(E) R - \{\lambda R_r V_\omega R\} = \\ &R_r - R_r \sigma(E) R - \lambda R_r V_\omega R_r \\ &\quad + \lambda R_r V_\omega R_r \sigma(E) R + \lambda^2 R_r V_\omega R_r V_\omega R, \end{aligned} \quad (3.9)$$

where the term in the curled brackets is the one we expanded according to the stopping rule. Note that the penultimate term is of order 3. It is not difficult to see that for a general  $N$  we get:

**Lemma 3.1.** *For any integer  $N$  we have a decomposition (used in Lemma 1.1)*

$$\begin{aligned} R(x, y) &= \sum_{z \in \mathbb{Z}^3} \left( \sum_{l=0}^{N-1} A'_l(x, z) R_r(z, y) + A'_N(x, z) R(z, y) + B_N(x, z) R(z, y) \right) \\ &= \sum_{l=0}^{N-1} A_l(x, y) + \sum_{z \in \mathbb{Z}^3} \tilde{A}_N(x, z) R(z, y), \end{aligned} \quad (3.10)$$

where  $A'_0(x, z) = \delta_{|x-z|}$ ,  $A'_l(x, z)$  is a summation over all possible terms of the type

$$\sum_{\theta, x_j \in \mathbb{Z}^3; j=1, \dots, n} R_r(x, x_1)\theta(x_1)R_r(x_1, x_2)\theta(x_2)\dots R_r(x_n, z)\theta(z) \quad (3.11)$$

which are of the order  $l > 0$ , while

$$B_N(x, z) = -\sigma(E) \sum_{w \in \mathbb{Z}^3} A'_{N-1}(x, w)R_r(w, z). \quad (3.12)$$

The quantities  $A_l$  and  $\tilde{A}_N$  are defined as

$$A_l(x, y) = \sum_{z \in \mathbb{Z}^3} A'_l(x, z)R_r(z, y), \quad \tilde{A}_N(x, y) = A'_N(x, y) + B_N(x, y).$$

*Proof.* Note first that it follows from (3.11, 3.12) that

$$A_{N+1}(x, y) = B_N(x, y) - \lambda \sum_{z, w \in \mathbb{Z}^3} A'_N(x, z)R_r(z, w)V_\omega(w)R(w, y). \quad (3.13)$$

We now prove (3.10) by induction: The base of induction,  $N = 0, 1$ , gives equalities

$$\begin{aligned} R(x, y) &= R(x, y), \\ R(z, y) &= R_r(z, y) - \sum_{w \in \mathbb{Z}^3} \left( \lambda R_r(z, w)V_\omega(w)R(w, y) + \sigma R_r(z, w)R(w, y) \right). \end{aligned}$$

Suppose that (3.10) holds for  $N$ , then

$$\begin{aligned} R(x, y) &= \sum_{z \in \mathbb{Z}^3} \left( \sum_{l=0}^{N-1} A'_l(x, z)R_r(z, y) + A'_N(x, z)R(z, y) + B_N(x, z)R(z, y) \right) \\ &= \sum_{z \in \mathbb{Z}^3} \left( \sum_{l=0}^{N-1} A'_l(x, z)R_r(z, y) + A'_N(x, z)R_r(z, y) + B_N(x, z)R(z, y) \right) \\ &\quad - \sum_{z, w \in \mathbb{Z}^3} A'_N(x, z) \left( \lambda R_r(z, w)V_\omega(w)R(w, y) + \sigma R_r(z, w)R(w, y) \right) \\ &= \sum_{z \in \mathbb{Z}^3} \left( \sum_{l=0}^N A'_l(x, z)R_r(z, y) + A'_{N+1}(x, z)R(z, y) + B_{N+1}(x, z)R(z, y) \right), \quad (3.14) \end{aligned}$$

where in the last line we used (3.12, 3.13).  $\square$

Such a stopping procedure guarantees cancelation of tadpoles and bullets in the following sense:

**Lemma 3.2.** *We have*

$$\begin{aligned} \mathbb{E} A_l^2(x, y) &= \lambda^{2l} \sum_{m=1}^l \sum'_{\{S_j\}_{j=1}^m} \sum_{(\mathbb{Z}^3)^{2l}} R_r(x, x_1)R_r(x, x_{l+2}) \prod_{i \in \Upsilon_{l,l}} R_r(x_i, x_{i+1}) \\ &\quad \times \prod_{j=1}^m c_{|S_j|} \delta(x_{S_j}), \quad (3.15) \end{aligned}$$

with  $\sum_{(\mathbb{Z}^3)^{2l}}$  standing for summation over  $\mathbb{Z}^3$  of all variables  $x_j$  with  $j \in \Upsilon_{l,l}$ ,  $x_{l+1} = x_{2l+2} = y$ , and where  $\sum'$  denotes summation over all possible partitions of  $\Upsilon_{l,l}$  which do not contain gates.

Equivalently, in the momentum representation

$$\begin{aligned} \mathbb{E} A_l^2(x, y) &= \lambda^{2l} \int e^{i\alpha} \frac{dp_{l+1}}{e(p_{l+1}) + E^*} \frac{dp_{2l+2}}{e(p_{2l+2}) + E^*} \\ &\times \prod_{t \in \Upsilon_{l,l}} \frac{dp_t}{e(p_t) + E^*} \sum_{m=1}^l \sum_{\pi=\{S_j\}_{j=1}^m} \prod_{j=1}^m c_{|S_j|} \delta\left(\sum_{i \in S_j} p_i - p_{i+1}\right), \quad (3.16) \end{aligned}$$

where

$$\alpha := 2\pi\{-(p_1 + p_{l+2}) \cdot x + (p_{l+1} + p_{2l+2}) \cdot y\}.$$

*Remark:* In order to obtain (3.16) from (3.15) one uses

$$R_r(z, w) = \int_{\mathbb{T}^3} e^{i2\pi(z-w)p} \frac{d^3p}{e(p) + E - \sigma(E)} = \int_{\mathbb{T}^3} e^{i2\pi(z-w)p} \frac{d^3p}{e(p) + E^*}.$$

*Proof.* Let us introduce some extra notation. Let  $\Pi_{l,l;k}$  denote the set of partitions of  $\Upsilon_{l,l}$  into disjoint subsets  $S_j$  such that each of these partitions contains exactly  $k$  tadpoles (that is there are  $k$  subsets  $S_j$  of the form  $S_j = \{i, i+1\}$ ). For every partition  $\Pi_{l,l;k} \ni \pi_k = \{S_j\}_{j=1}^m$  we will denote by  $\Upsilon(\pi_k)$  a subset of  $\Upsilon_{l,l}$  which consists of the indices that differ from the set  $\Upsilon^c(\pi_k) := \Upsilon_{l,l} \setminus \Upsilon(\pi_k)$  of the gate's indices associated with  $\pi_k$  (so  $\text{card}(\Upsilon(\pi_k)) = 2l - 2k$ ). Let  $\hat{\pi}_k = \{S_j\}_{\{j: S_j \subseteq \Upsilon(\pi_k)\}}$ . Let  $\nu(v)$  be the connected segment of the set of the indices  $\Upsilon^c(\pi_k)$  which contains index  $v$ , that is

$$v \in \nu(v) \subseteq \Upsilon^c(\pi_k)$$

and  $\nu(v) = \{i, i+1, \dots, h-1, h\}$ , with  $i-1, h+1 \in \Upsilon(\pi_k)$ . Let  $d(v)$  denote the position of index  $v$  with respect to  $\nu(v)$ , for example, if  $\nu(4) = \{3, 4, 5, 6, 7, 8\}$ , then  $d(4) = 2$ . We define  $\hat{\prod}_{v \in \Upsilon^c(\pi_k)}$  to be a product over such  $v \in \Upsilon^c(\pi_k)$  that  $d(v) \bmod 2 = 1$ . For instance, if  $\Upsilon^c(\pi_k) = \{1, 2, 4, 5, 6, 7\}$ , the product will run over the variables 1, 4, 6.

We can now express  $A_l(x, y)^2$  as

$$\begin{aligned} A_l^2(x, y) &= \sum_{k=0}^l \lambda^{2l-2k} (-\sigma(E))^k \sum_{\pi_k \in \Pi_{l,l;k}} \sum_{(\mathbb{Z}^3)^{2l}} R_r(x, x_1) R_r(x, x_{l+2}) \\ &\times \prod_{i \in \Upsilon(\pi_k)} V_\omega(x_i) R_r(x_i, x_{i+1}) \hat{\prod}_{j \in \Upsilon^c(\pi_k)} R_r(x_{j+1}, x_{j+2}) \delta_{|x_{j+1} - x_j|}. \quad (3.17) \end{aligned}$$

Note that here the index  $k$  corresponds to the number of bullets in the corresponding contribution, and is not related to the number of the tadpoles (which will show up as index  $k'$  below once we undertake the expectation over disorder).

On the other hand, since  $c_2 = 1$ , one obtains an identity

$$\begin{aligned} \sum_{m=1}^N \sum_{\pi=\{S_j\}_{j=1}^m} \prod_{j=1}^m c_{|S_j|} \delta(x_{S_j}) \\ = \sum_{k'=0}^N \sum_{m=1}^{N-k'} \sum_{\hat{\pi}_{k'}=\{S_j\}_{j=1}^m} \prod_{j=1}^m c_{|S_j|} \delta(x_{S_j}) \hat{\prod}_{v \in \Upsilon^c(\pi_{k'})} \delta_{|x_{v+1} - x_v|}. \quad (3.18) \end{aligned}$$

At this point, let us first compute the expectation of  $k = 0$  part of the summation in (3.17):

$$\begin{aligned}
& \lambda^{2l} \sum_{\pi_0 \in \Pi_{l,l;0}} \sum_{(\mathbb{Z}^3)^{2l}} R_r(x, x_1) R_r(x, x_{l+2}) \mathbb{E} \left[ \prod_{i \in \Upsilon(\pi_0)} V_\omega(x_i) R_r(x_i, x_{i+1}) \right] \\
&= \lambda^{2l} \sum_{k'=0}^l \sum_{m=1}^{l-k'} \sum_{\hat{\pi}_{k'} = \{S_j\}_{j=1}^m} \sum_{(\mathbb{Z}^3)^{2l}} R_r(x, x_1) R_r(x, x_{l+2}) \prod_{j=1}^m c_{|S_j|} \delta(x_{S_j}) \\
&\quad \times \prod_{i \in \Upsilon(l,l)} R_r(x_i, x_{i+1}) \prod_{v \in \Upsilon^c(\pi_{k'})} \delta_{|x_{v+1} - x_v|} \\
&= \sum_{k'=0}^l \lambda^{2l-2k'} (\sigma(E))^{k'} \sum_{m=1}^{l-k'} \sum_{\hat{\pi}_{k'} = \{S_j\}_{j=1}^m} \sum_{(\mathbb{Z}^3)^{2l}} R_r(x, x_1) R_r(x, x_{l+2}) \\
&\quad \times \prod_{j=1}^m c_{|S_j|} \delta(x_{S_j}) \prod_{i \in \Upsilon(\pi_{k'})} R_r(x_i, x_{i+1}) \prod_{j \in \Upsilon^c(\pi_{k'})} R_r(x_{j+1}, x_{j+2}) \delta_{|x_{j+1} - x_j|}, \quad (3.19)
\end{aligned}$$

where we have used  $\lambda^2 R_r(z, z) = \sigma(E)$  for all  $z \in \mathbb{Z}^3$ .

More generally, we have an equality

$$\begin{aligned}
& \lambda^{2l-2k} (-\sigma(E))^k \sum_{\pi_k \in \Pi_{l,l;k}} \sum_{(\mathbb{Z}^3)^{2l}} R_r(x, x_1) R_r(x, x_{l+2}) \\
&\quad \times \mathbb{E} \left[ \prod_{i \in \Upsilon(\pi_k)} V_\omega(x_i) R_r(x_i, x_{i+1}) \right] \prod_{j \in \Upsilon^c(\pi_k)} R_r(x_{j+1}, x_{j+2}) \delta_{|x_{j+1} - x_j|} \\
&= \sum_{k'=0}^{l-k} \binom{k'+k}{k'} \lambda^{2l-2k-2k'} (-\sigma(E))^k (\sigma(E))^{k'} \\
&\quad \times \sum_{m=1}^{l-k-k'} \sum_{\hat{\pi}_{k'+k} = \{S_j\}_{j=1}^m} \sum_{(\mathbb{Z}^3)^{2l}} R_r(x, x_1) R_r(x, x_{l+2}) \\
&\quad \times \prod_{j=1}^m c_{|S_j|} \delta(x_{S_j}) \prod_{i \in \Upsilon(\pi_{k'+k})} R_r(x_i, x_{i+1}) \prod_{j \in \Upsilon^c(\pi_{k'+k})} R_r(x_{j+1}, x_{j+2}) \delta_{|x_{j+1} - x_j|}. \quad (3.20)
\end{aligned}$$

Therefore, the expectation of the rhs of (3.17) is given by the formula

$$\begin{aligned}
\mathbb{E} A_l^2(x, y) &= \sum_{\beta=0}^l \lambda^{2l-2\beta} \sigma^\beta(E) \sum_{k,k': \{k+k'=\beta\}} (-1)^k \binom{k'+k}{k'} \\
&\quad \times \sum_{m=1}^{l-\beta} \sum_{\hat{\pi}_\beta = \{S_j\}_{j=1}^m} \sum_{(\mathbb{Z}^3)^{2l}} R_r(x, x_1) R_r(x, x_{l+2}) \\
&\quad \times \prod_{j=1}^m c_{|S_j|} \delta(x_{S_j}) \prod_{i \in \Upsilon(\pi_\beta)} R_r(x_i, x_{i+1}) \prod_{j \in \Upsilon^c(\pi_\beta)} R_r(x_{j+1}, x_{j+2}) \delta_{|x_{j+1} - x_j|}, \quad (3.21)
\end{aligned}$$

and the only non vanishing contribution comes from  $\beta = 0$ , since

$$\sum_{k,k':\{k+k'=\beta\}} (-1)^{k'} \binom{k'+k}{k'} = (1-1)^\beta,$$

hence (3.15).  $\square$

At this point we have to introduce additional notation (borrowed from [6]):

**Definition 1.** We consider products of delta functions with arguments that are linear combinations of the momenta  $\{p_1, p_2, \dots, p_{2n+2}\}$ . Two products of such delta functions are called *equivalent* if they determine the same affine subspace of  $\mathbb{T}^{2n+2} = \{p_1, p_2, \dots, p_{2n+2}\}$ .

One can obtain new delta functions from the given ones, by taking linear combinations of their arguments. In particular, we can obtain identifications of momenta.

**Definition 2.** The product of delta functions  $\Delta_\pi$  *forces* a new delta function  $\delta(\sum_j a_j p_j)$ , if  $\sum_j a_j p_j = 0$  is an identity in the affine subspace determined by  $\Delta_\pi$ .

One can readily see that in the integrand of rhs of (3.16) one has a forced delta function  $\delta(p_1 - p_{l+1} + p_{l+2} - p_{2l+2})$ , hence

$$\begin{aligned} A_{n,E^*}(x-y) := \mathbb{E} A_n^2(x,y) &= \lambda^{2n} \int e^{-i2\pi(p_1+p_{n+2}) \cdot (x-y)} \prod_{t=1}^{2n+2} dp_t \frac{1}{e(p_t) + E^*} \\ &\times \sum_{m=1}^n \sum_{\pi=\{S_j\}_{j=1}^m} \prod_{j=1}^m c_{|S_j|} \delta\left(\sum_{i \in S_j} p_i - p_{i+1}\right). \end{aligned} \quad (3.22)$$

#### 4. PROOF OF LEMMA 1.1

Let us start with a remark that although the proof below is inspired (and closely follows) by work of Erdos and Yau [6], the geometry of energy surfaces plays no significant role here. In [6], the geometry of energy surfaces poses a central difficult problem because the energy parameter varies, and in particular assumes values in the bulk of the essential spectrum of the nearest neighbor Laplacian. However, in the situation discussed in this paper, the reference energy  $E^*$  is fixed, and lies below  $\inf(\sigma(-\Delta)) = 0$ .

**4.1. Reduction to the pairing case.** In order to estimate  $A_{n,E^*}(x-y)$  it suffices (up to the combinatorial factors) to consider the special case of partitions  $\pi$  that appears in (3.15), where  $\pi = \{S_j\}_{j=1}^n$  with  $\text{card } S_j = 2$  for each  $j$  - the so called pairing case. All other contributions are dominated by the corresponding pairing counterparts, as becomes transparent from the positivity of the free lattice Green function  $R_r(x,y)$ , cf. (1.13). Indeed, for any partition  $\pi = \{S_j\}_{j=1}^m$  choose an arbitrary subpartition  $\pi' = \{S'_j\}_{j=1}^m$  into pairs. Evidently,

$$\begin{aligned} \sum_{(\mathbb{Z}^3)^{2n}} R_r(x, x_1) R_r(x, x_{l+2}) \prod_{i \in \Upsilon_{l,l}} R_r(x_i, x_{i+1}) \prod_{j=1}^m \delta(x_{S_j}) \\ \leq \sum_{(\mathbb{Z}^3)^{2n}} R_r(x, x_1) R_r(x, x_{l+2}) \prod_{i \in \Upsilon_{l,l}} R_r(x_i, x_{i+1}) \prod_{j=1}^m \delta(x_{S'_j}), \end{aligned} \quad (4.1)$$

while the factor  $\prod_{j=1}^m |c_{|S_j|}|$  is bounded by  $(cn)^{2n+1}$ , see discussion in Subsection 3.1. Hence, if one gets some bound  $M$  on the pairing type contributions, the whole  $A_{n,E^*}(x-y)$  term can be rudely estimated as  $(2cn^2)^{2n+1}M$  (where we took into the account the number of the possible partitions).

**4.2. Feynman graphs.**  $A_{n,E^*}(x-y)$  is conveniently interpreted in terms of the so called Feynman graphs (the pseudograph, to be precise, since loops and multiple edges are allowed here). The graph, associated with particular partition  $\pi$  of  $\Upsilon_{n,n}$  is constructed according to the following rules (see Figure 1 and 2): We first draw two line segments, each containing  $n$  vertices (elements of  $\Upsilon_{n,n}$ ). The vertices are joined by directed edges (momentum lines) representing momenta:  $p_1, \dots, p_{n+1}$  and  $p_{n+2}, \dots, p_{2n+2}$ . To each line  $p_j$  we assign a propagator  $F(p_j)$ , with some given function  $F$ , save momentum lines  $p_1$  and  $p_{n+2}$ , which carry additional phases  $e^{-i2\pi p_1 \cdot (x-y)}$  and  $e^{-i2\pi p_{n+2} \cdot (x-y)}$ , respectively. For  $\pi = \{S_j\}_{j=1}^m$  we identify all vertices in each subset  $S_j$  as the same vertex (in Figure 1, the paired vertices are connected by dashed lines). Note that thanks to the existence of the forced delta

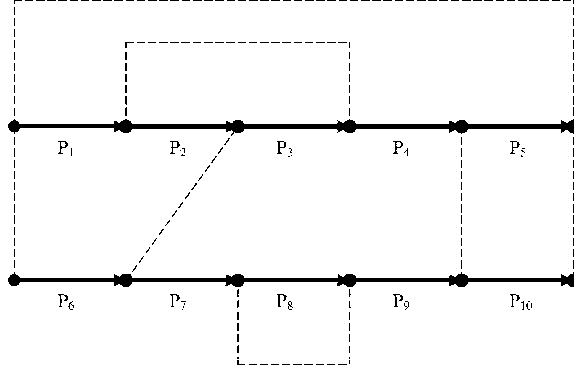


FIGURE 1. Construction of the Feynman graph, part I,  $n = 4$ . The corresponding delta functions are  $\delta(p_1 - p_2 + p_3 - p_4)$ ,  $\delta(p_4 - p_5 + p_9 - p_{10})$ ,  $\delta(p_2 - p_3 + p_6 - p_7)$ , and  $\delta(p_7 - p_9)$ . The last delta corresponds to the tadpole. Note that the sum of all momenta in the above delta functions gives a forced delta function  $\delta(p_1 - p_5 + p_6 - p_{10})$ , hence we can introduce the dashed lines connecting vertices 1, 6, 7, and 12, identifying them as a single vertex.

function  $\delta(p_1 - p_{l+1} + p_{l+2} - p_{2l+2})$ , we can identify vertices  $1, n, n+1, 2n$  as a single one, and therefore one can think about the closed graph (with special rules that apply for momentum lines  $p_1$  and  $p_{n+2}$ , mentioned above). To summarize, the outcome of this construction is a directed closed graph, which is called the Feynman graph associated with the partition  $\pi$ . The momenta in the graph satisfy the Kirchhoff's first law, that is the total momenta entering into each internal vertex add up to zero (if arrow faces outward the vertex, we count its momentum with a minus sign). A tadpole corresponds to the so-called 0-loop, that is some (directed)

line of the graph claims one vertex as its both endpoints. For a given Feynman graph  $G$ , one can choose a particularly useful expression for the product of delta functions  $\Delta_\pi$ . Choose any spanning tree of  $G$  which does not contain momentum lines  $p_1, p_{n+2}$ . The edges belonging to the spanning tree will be called the *tree* edges (momentum lines), and all the rest are the *loop* edges (since an addendum of any loop's momentum line creates a loop). Let us enumerate the tree variables as  $u_1, \dots, u_k$ , and loop variables as  $w_1, \dots, w_l$ , with say  $w_1 = p_1, w_2 = p_{n+2}$  (note that  $k + l = 2n + 2$ ). The number  $k$  of the tree momenta coincides with the number of the delta functions in  $\Delta_\pi$ . One can check (see e.g. [6]) that the product of delta

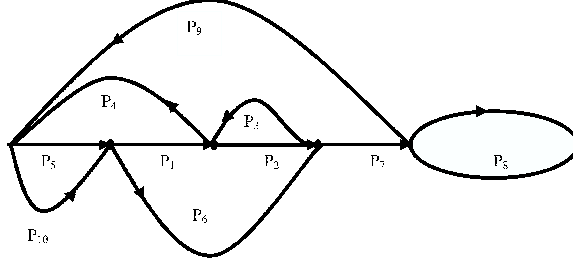


FIGURE 2. Construction of the Feynman graph, part II: Identification of the vertices. The tadpole corresponds here to 0-loop.

functions  $\Delta_\pi$  is equivalent to

$$\prod_{i=1}^k \delta(u_i - \sum_{j=1}^l a_{ij} w_j), \quad (4.2)$$

with

$$a_{ij} := \begin{cases} \pm 1 & \text{loop is created by adding } w_j \text{ to the spanning tree contains } u_i \\ 0 & \text{otherwise} \end{cases}.$$

The choice of the sign depends on the mutual orientation of  $u_i$  and  $w_j$ .

**4.3. Exponential decay.** We first want to establish the exponential decay of  $A_{n,E^*}(x)$  that appears (3.22) from the simple analytic argument, and then in the next section obtain the bound on  $A_{n,E^*}(0)$ .

Specifically, we want to show that for a general value of  $n$ ,

$$A_{n,E^*}(x) \leq e^{-|x|\sqrt{E^*/3}} A_{n,E^*/2}(0). \quad (4.3)$$

Indeed, for any given  $x \in \mathbb{Z}^3$  let's choose  $\gamma \in \{1, 2, 3\}$  such that

$$|x \cdot e_\gamma| = \max_{i \in \{1, 2, 3\}} |x \cdot e_i|. \quad (4.4)$$



Then  $|x \cdot e_\gamma| \geq |x|/\sqrt{3}$ . In order to obtain the exponential decay of  $A_{n,E^*}(x)$  we first perform the integration in the rhs of (3.22) over the tree momenta, using (4.2). Let us use the shorthand notation  $\sum_\pi$  for a sum over all possible partitions in (4.2),  $c_\pi$  for a product of the corresponding  $c_{Sj}$ , and  $r_\pi$  will denote the number of the delta functions containing the loop momentum  $w_1$  in the  $\pi$ 's partition. We get

$$A_{n,E^*}(x) = \lambda^{2n} \sum_\pi c_\pi \int dw_1 \prod_{i=1}^{r_\pi} \frac{1}{e(w_1 + q_i) + E^*} e^{-i2\pi w_1 \cdot x} \cdot \int \prod_{t \in \Phi'} dp_t e^{-i2\pi w_2 \cdot x} \prod_{i=r_\pi+1}^{2n+2} \frac{1}{e(q_i) + E^*}, \quad (4.5)$$

where  $\Phi'$  is a set of all loop variables in the partition  $\pi$ , except for  $w_1$ . The variable  $q_k$  is some linear combination of the loop variables in  $\Phi'$ , for  $k = 1, \dots, 2n+2$ . We want to obtain a bound on the integral over  $w_1$  variable. Note that

$$\begin{aligned} & \int dp \prod_{i=1}^r \frac{1}{e(p + q_i) + E^*} e^{-i2\pi p \cdot x} \\ &= \int dp' e^{-i2\pi(p \cdot x - p \cdot e_\gamma x \cdot e_\gamma)} \int_{-1/2}^{1/2} d(p \cdot e_\gamma) \prod_{i=1}^r \frac{1}{e(p + q_i) + E^*} e^{-i2\pi(p \cdot e_\gamma x \cdot e_\gamma)}, \end{aligned} \quad (4.6)$$

where  $\int dp'$  stands for integration over components of  $p$  orthogonal to  $e_\gamma$ . Without loss of generality, let us assume that  $x \cdot e_\gamma > 0$ . It is easy to check that the integrand as a function of  $p \cdot e_\gamma$  is 1-periodic, analytic inside the rectangle formed by the points

$$\{-1/2; -1/2 + i\sqrt{E^*}/5; 1/2 + i\sqrt{E^*}/5; 1/2\}$$

for sufficiently small  $E^*$ : Indeed, we have

$$\operatorname{Re} e(p + q + i\epsilon e_\gamma) + E^* \geq e(p + q) + E^*/2$$

uniformly in  $q$ , provided

$$0 \leq \epsilon \leq \frac{\sqrt{E^*}}{3}$$

for small  $E^*$ , where we have used  $\sin(a + ib) = \sin a \cosh b + i \cos a \sinh b$ . Moreover, the periodicity implies that the integrals over the vertical segments coincide:

$$\begin{aligned} & \int_{-1/2}^{-1/2 + i\sqrt{E^*}/(2\pi)} d(p \cdot e_\gamma) \prod_{i=1}^r \frac{1}{e(p + q_i) + E^*} e^{-i2\pi(p \cdot e_\gamma x \cdot e_\gamma)} \\ &= \int_{1/2}^{1/2 + i\sqrt{E^*}/(2\pi)} d(p \cdot e_\gamma) \prod_{i=1}^r \frac{1}{e(p + q_i) + E^*} e^{-i2\pi(p \cdot e_\gamma x \cdot e_\gamma)}. \end{aligned} \quad (4.7)$$

Therefore

$$\begin{aligned}
& \left| \int_{-1/2}^{1/2} d(p \cdot e_\gamma) \prod_{i=1}^r \frac{1}{e(p + q_i) + E^*} e^{-i2\pi(p \cdot e_\gamma x \cdot e_\gamma)} \right| \\
&= \left| \int_{-1/2+i\sqrt{E^*}/(2\pi)}^{1/2+i\sqrt{E^*}/(2\pi)} d(p \cdot e_\gamma) \prod_{i=1}^r \frac{1}{e(p + q_i) + E^*} e^{-i2\pi(p \cdot e_\gamma x \cdot e_\gamma)} \right| \\
&\leq e^{-x \cdot e_\gamma \sqrt{E^*}} \int_{-1/2}^{1/2} d(p \cdot e_\gamma) \prod_{i=1}^r \frac{1}{e(p + q_i) + E^*/2} \\
&\leq e^{-|x|\sqrt{E^*}/3} \int_{-1/2}^{1/2} d(p \cdot e_\gamma) \prod_{i=1}^r \frac{1}{e(p + q_i) + E^*/2}, \quad (4.8)
\end{aligned}$$

hence (4.3).

**4.4. Bound on the pairing type diagrams.** Note that for any  $p \in \mathbb{T}^3$  we have an elementary estimate

$$e(p) = 2 \sum_{i=1}^3 \sin^2(\pi p \cdot e_i) > 2 \sin^2\left(\frac{\pi|p|}{\sqrt{3}}\right).$$

On the other hand, by Jordan's inequality  $\sin^2\left(\frac{\pi|p|}{\sqrt{3}}\right) \geq \frac{4p^2}{3}$  for any  $p \in \mathbb{T}^3$ . Therefore,

$$(e(p) + E^*)^{-1} \leq C(p^2 + E^*)^{-1} \quad (4.9)$$

for any  $p \in \mathbb{T}^3$ , and

$$\begin{aligned}
A_{n,E^*}(0) &\leq \lambda^{2n} \int \prod_{t=1}^{2n+2} dp_t \frac{C}{p_t^2 + E^*} \\
&\quad \times \sum_{m=1}^n \sum_{\pi=\{S_j\}_{j=1}^m}' \left( \prod_{j=1}^m |c_{|S_j|}| \right) \delta\left(\sum_{i \in S_j} p_i - p_{i+1}\right). \quad (4.10)
\end{aligned}$$

Now consider the pairing type partition  $\pi$ . For such a partition all  $c_{|S_j|} = 1$ . It is natural to rescale variables,  $p_t = \sqrt{E^*} q_t$ , to get

$$\begin{aligned}
& \int \prod_{t=1}^{2n+2} dp_t \frac{C}{p_t^2 + E^*} \delta\left(\sum_{i \in S_j} p_i - p_{i+1}\right) \\
&= (E^*)^{-\frac{n}{2}+1} \int \prod_{t=1}^{2n+2} dq_t \frac{C}{q_t^2 + 1} \prod_{j=1}^n \delta\left(\sum_{i \in S_j} q_i - q_{i+1}\right), \quad (4.11)
\end{aligned}$$

where we used the scaling  $\delta(ap) = a^{-3}\delta(p)$  in three dimensions (and the fact that there are  $n$  delta functions involved in the pairing case). Let us note that  $n$  is equal to the number of loop momenta and equal to the number of vertices minus 1. Each integration now runs over the torus  $(E^*)^{-1/2}\mathbb{T}^3$ , concisely denoted as  $\int'$ . We can

only increase the right hand side by replacing it with

$$\begin{aligned}
& C^{2n+2} (E^*)^{-\frac{n}{2}+2} \int \prod_{t=1}^{2n+2} dq_t \frac{\ln^4((E^*)^{-1} + 1)}{(q_t^2 + 1) \ln^4(q_t^2 + 2)} \prod_{j=1}^n \delta(\sum_{i \in S_j} q_i - q_{i+1}) \\
& < \tilde{C}^{2n+2} \frac{\ln^{8n+8} E^*}{(E^*)^{\frac{n}{2}-1}} \int_{(\mathbb{R}^3)^{2n+2}} \prod_{t=1}^{2n+2} dq_t \frac{1}{(q_t^2 + 1) \ln^4(q_t^2 + 2)} \prod_{j=1}^n \delta(\sum_{i \in S_j} q_i - q_{i+1}),
\end{aligned} \tag{4.12}$$

where  $\delta(\cdot)$  in the last line stands for the standard Dirac delta distribution on  $\mathbb{R}^3$ .

At this point we are going to switch to the (Euclidean) Feynman graph representation, as described in Subsection 4.2, with the propagator

$$F(q_j) := \frac{1}{(q_j^2 + 1) \ln^4(q_j^2 + 2)}.$$

Let us define the *value*  $|G|$  of the graph  $G$  (characterized by its partition  $\pi = \{S_j\}_{j=1}^n$ ) as

$$|G| = \int_{(\mathbb{R}^3)^{2n+2}} \prod_{t=1}^{2n+2} dq_t \frac{1}{(q_t^2 + 1) \ln^4(q_t^2 + 2)} \prod_{j=1}^n \delta(\sum_{i \in S_j} q_i - q_{i+1}). \tag{4.13}$$

If partition  $\pi$  is tadpole-free (and those are only partitions that enter into (3.22) thanks to the self energy renormalization), the corresponding Feynman diagram has an important property of being (logarithmically) superficially convergent. The value of such graphs is controlled by the following theorem (an adaptation of Theorem A1 in [7] to the case in hand):

**Theorem 3** (Bound on superficially convergent Euclidean Feynman graphs, [7]). *If Feynman graph satisfies assumptions A1-A3 below, then its value is bounded by  $K^n$ , with some generic constant  $K$ .*

This result yields that the contribution from a pairing type partition to  $A_{n,E^*}(0)$  is bounded by

$$E^* \left( C \ln^9 E^* \frac{\lambda^2}{\sqrt{E^*}} \right)^n \tag{4.14}$$

- the key estimate of this subsection.

The subsequent assumptions require the introduction of some additional notation. Let a *subgraph*  $G' \subseteq G$  be a subset of the lines of  $G$ . The vertices of  $G'$  are the end points of lines of  $G'$  and an external line of  $G'$  is an edge of  $G \setminus G'$  which is hooked to a vertex of  $G'$ .

Let us denote by  $\Lambda(G')$  the number of loop edges of  $G'$  (defined in Subsection 4.2), by  $E(G')$  the number of external edges of  $G'$ , by  $I(G')$  the number of internal lines of  $G'$ , and by  $N(G')$  the number of vertices in  $G'$ .

Two subgraphs  $F, F'$  are *disjoint* if they have no line and no vertex in common. They *overlap* if they are not disjoint and do not satisfy an inclusion relation ( $F \subseteq F'$  or  $F' \subseteq F$ ). A *forest*  $\mathbb{F}$  is a set of non-overlapping connected subgraphs.

A subgraphs  $F$  is said to be *one line reducible* (OLR) if there exists a line of  $F$  such that its removal increases the number of connected components of  $F$ . A connected subgraph which is not OLR is called *proper*.

**A1** *The lines*

Each line has a propagator of the form

$$\frac{1}{(q^2 + 1) \ln^4(q^2 + 2)}.$$

**A2** *Superficial convergence*

For any connected subgraph  $G' \subseteq G$  a superficial degree of divergence  $div(G')$  is given by

$$div(G) = 3\Lambda(G) - 2I(G)$$

For any connected subgraph  $G' \subseteq G$  a logarithmical superficial degree of divergence  $l - div(G')$  is given by

$$l - div(G') = \Lambda(G') - 4I(G').$$

A graph  $G$  is called *superficially convergent* if any connected subgraph  $G'$  of  $G$  satisfies either

$$div(G') < -2\epsilon E(G')$$

or

$$div(G') = 0 \text{ and } l - div(G') \leq -\epsilon.$$

**A3** *div = 0 Forests*

- (i) There exists a constant  $C$  such that the number of proper  $div = 0$  forests of  $G$  (i.e. forests consisting of proper subgraphs  $G'$  with  $div(G') = 0$ ) is bounded by  $C^{L(G)}$ .
- (ii) There exists a constant  $C'$  such that for every connected  $G' \subseteq G$  with  $div(G') = 0$ ,  $G'$  has at most  $C'$  external vertices.

*Remark:* Theorem A1 in [7] is a much more general result than the one presented here. It contains two additional assumptions which are irrelevant in our context (namely HA.2–HA.3 in [7]). We also adapted the various definitions from [7] (such as a superficial degree of divergence) to the concrete situation discussed in this paper.

The rest of this subsection is devoted to the validation of the assumptions A2–A3 of Theorem 3 in the present context. A3 is met since all superficially divergent subgraphs in our situation (in fact there is only one divergent subgraph  $F$ , introduced below on Fig. 3) are also divergent in  $\phi_4^4$  theory, where this assumption holds true with  $C = 8$  ([4]). The constant  $C'$  (which in our context corresponds to the number of external lines of the aforementioned graph  $F$ ) is equal to 2.

We now want to establish the validity of A2, with  $\epsilon = \frac{1}{10}$ :

Let us note that for a pairing partition, the degree of each internal vertex is 4, that is we have 4-regular directed graph. For such a graph any spanning tree contains  $n$  edges, and  $n + 2$  loops accordingly.

Since  $G$  is 4-regular, it is easy to see that for any connected subgraph  $G'$

$$I(G') \leq \frac{4N(G') - E(G')}{2}; \quad \Lambda(G') + N(G') - 1 = I(G'), \quad (4.15)$$

where the latter relation follows from the fact that the spanning tree for  $G'$  contains  $N(G') - 1$  lines, and the rest of the internal lines can be thought of as loops. Hence

$$\begin{aligned} div(G') &= -\Lambda(G') + 2(2\Lambda(G') - I(G')) \\ &= -\Lambda(G') + 2(I(G') + 2 - 2N(G')) \leq 4 - E(G') - \Lambda(G') \end{aligned} \quad (4.16)$$

and

$$l - \text{div}(G') = (\Lambda(G') - I(G')) - 3I(G') = 1 - N(G') - 3I(G') \leq -4 \quad (4.17)$$

for any subgraph  $G'$  of  $G$  with  $N(G') \geq 2$ . Since  $N(G') = 1$  corresponds to the tadpole, which is not allowed, we conclude than is that all relevant subgraphs are logarithmically convergent.

For the whole graph  $G$  we have  $\text{div}(G) = 3(n+2) - 2(n+2) < -n/4$  for  $n > 2$ , while  $\text{div}(G) = 0$  for  $n = 2$ . Since  $2E(G') \bmod 4 = 0$  for any subgraph  $G'$  of the 4-regular graph  $G$ , and  $E(G') \neq 0$  unless  $G' = G$ , we deduce from (4.15) and (4.16) that for any Feynman graph  $G$  corresponding to (4.12) with  $n \geq 2$  the only possible proper subgraphs  $G'$  with  $\text{div}(G') \geq 0$  are:

- (1)  $N(G') = 1$ ,  $\Lambda(G') = 1$ ,  $E(G') = 2$  - a 0-loop, that is a tadpole. For the 0-loop  $\text{div}(G') = 1$ , but on the other hand, the tadpoles are prohibited in our partition.
- (2)  $N(G') = 2$ ,  $\Lambda(G') = 2$ ,  $E(G') = 2$  - either a pair of tadpoles connected by an edge or a graph  $F$ , shown on Figure 3 (with two external edges omitted). For the latter graph we have  $\text{div}(F) = 0$ , hence  $F$  is superficially convergent as well.

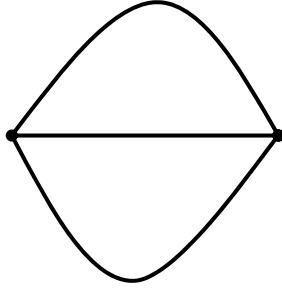


FIGURE 3. Graph  $F$ .

Since it is clear from (4.16) that  $\text{div}(G') \leq -E(G')/5$  for  $E(G') \geq 5$  while  $\text{div}(G') \leq -1 \leq -E(G')/5$  for  $E(G') \leq 5$  (with the exception of the graph  $F$ ), we conclude that all tadpole-free pairings are indeed superficially convergent.

**4.5. Bounds (1.11–1.15).** We deduce from (4.14), (4.1) and discussion thereafter that

$$A_{n,E^*}(0) \leq (4n)! E^* \left( C \ln^9 E^* \frac{\lambda^2}{\sqrt{E^*}} \right)^n,$$

and (1.11) follows now from (4.3).

To get (1.12) note that it follows from Lemma 3.1 that

$$\tilde{A}_N = A_N \left( -\frac{1}{2} \Delta + E^* \right) - \sigma(E) A_{N-1} R_r.$$

We therefore obtain

$$\begin{aligned}
\mathbb{E} |\tilde{A}_N(x, y)| &\leq \sum_{z \in \mathbb{Z}^3} \left\{ \left( \mathbb{E} |A_N(x, z)|^2 \right)^{1/2} \cdot \left| \left( -\frac{1}{2} \Delta + E^* \right)(z, y) \right| \right. \\
&\quad \left. + \sigma(E) \left( \mathbb{E} |A_{N-1}(x, z)|^2 \right)^{1/2} \cdot R_r(z, y) \right\} \\
&< 7 \sum_{z \in \mathbb{Z}^3: |z-y| \leq 2} \left( \mathbb{E} |A_N(x, z)|^2 \right)^{1/2} \\
&\quad + C \sum_{z \in \mathbb{Z}^3} \sigma(E) \left( \mathbb{E} |A_{N-1}(x, z)|^2 \right)^{1/2} \cdot \frac{1}{|z-y|+1} e^{-\sqrt{2E^*}|z-y|},
\end{aligned}$$

with some generic constant  $C$ , provided  $E^* \leq 1$  and where we used the bound (1.14) on the free Green function  $R_r(z, y)$ .

It is clear from (1.11) that the first contribution is bounded by

$$C' \sqrt{(4N)! E^*} \left( C \ln^9 E^* \frac{\lambda^2}{\sqrt{E^*}} \right)^{N/2} e^{-\sqrt{\frac{E^*}{12}}|x-y|},$$

while the second one is bounded by

$$\begin{aligned}
C' P(\lambda, E^*) \sum_{z \in \mathbb{Z}^3} \frac{1}{|z-y|+1} e^{-\sqrt{\frac{E^*}{12}}|x-z|} e^{-\sqrt{2E^*}|z-y|} \\
< C' P(\lambda, E^*) e^{-\sqrt{\frac{E^*}{12}}|x-y|} \sum_{z \in \mathbb{Z}^3} \frac{1}{|z-y|+1} e^{-\sqrt{2E^*}(1-1/\sqrt{24})|z-y|} \\
< \frac{\tilde{C}}{E^*} P(\lambda, E^*) e^{-\sqrt{\frac{E^*}{12}}|x-y|},
\end{aligned}$$

with

$$P(\lambda, E^*) := \lambda^2 \sqrt{(4N-4)! E^*} \left( C \ln^9 E^* \frac{\lambda^2}{\sqrt{E^*}} \right)^{N/2-1},$$

hence (1.12).

The positivity of  $A_0(x, y)$  in (1.13) is immediate from the positivity of  $e^{t\Delta}(x, y)$  for all non negative values of  $t$ , which in turn is obvious, since all off diagonal matrix elements of the Laplacian are positive, and its diagonal part is proportional to the unity operator.

The upper bound in (1.13) is the standard estimate, see e.g. [12], it also follows from (1.14). The bound (1.14) for the free Green's function is known as well, see e.g. [9].

The convenient way to establish bound (1.15) is in the coordinate representation: Since  $\mathbb{E}(V_\omega(x)V_\omega(y)) = \delta_{|x-y|}$  one checks that

$$\mathbb{E} A_1^2(x, y) = \lambda^2 \sum_{z \in \mathbb{Z}^3} R_r^2(x, z) R_r^2(z, y).$$

Now we use (1.14) to bound the rhs from above as

$$\begin{aligned} \lambda^2 \sum_{z \in \mathbb{Z}^3} \frac{1}{|x-z|^2+1} \frac{1}{|y-z|^2+1} e^{-2\sqrt{2E^*}(|x-z|+|y-z|)} \\ \leq \lambda^2 e^{-2\sqrt{2E^*}|x-y|} \sum_{z \in \mathbb{Z}^3} \frac{1}{|x-z|^2+1} \frac{1}{|y-z|^2+1} \\ \leq \frac{C\lambda^2}{|x-y|+1} e^{-2\sqrt{2E^*}|x-y|}. \end{aligned} \quad (4.18)$$

#### APPENDIX A. PROPERTIES OF THE SOLUTION OF (1.3)

It is instructive to rewrite (1.3) as

$$E = E^* + \lambda^2 \int_{\mathbb{T}^3} \frac{d^3p}{e(p) + E^*}. \quad (A.1)$$

$E$  then is a well defined function of  $E^*$  on the ray  $E^* \in [0, \infty)$ , with

$$E(0) = \lambda^2 \int_{\mathbb{T}^3} \frac{d^3p}{e(p)}.$$

Differentiation of (A.1) with respect to  $E^*$  gives

$$E'(E^*) = 1 - \lambda^2 \int_{\mathbb{T}^3} \frac{d^3p}{(e(p) + E^*)^2}. \quad (A.2)$$

Monotonicity of the integral on the rhs of the above equation for  $E^* \in (0, \infty)$  implies that there exists exactly one extremum of  $E$  (namely minimum) for such values of  $E^*$ . In particular, the function  $E = E(E^*)$  is invertible for  $E > E(0)$ . For any  $E^* > 0$  we have an estimate

$$\int_{\mathbb{T}^3} \frac{d^3p}{(e(p) + E^*)^2} \leq C(E^*)^{-1/2},$$

which follows from (4.9) and the extension of the domain of integration to the whole  $\mathbb{R}^3$ . We infer that

$$E(E^*) = \int_0^{E^*} E'(t)dt + E(0) \geq \lambda^2 \int_{\mathbb{T}^3} \frac{d^3p}{e(p)} + E^* - C\lambda^2 \int_0^{E^*} t^{-1/2}dt,$$

hence  $E(E^*) > E(0)$  for  $E^* > 4C^2\lambda^4$ , so there exists an inverse function  $E^* = E^*(E)$  in this range. On the other hand, note that  $e(p) \leq p^2/2$ , hence

$$\int_{\mathbb{T}^3} \frac{d^3p}{(e(p) + E^*)^2} \geq \int_{\mathbb{T}^3} \frac{d^3p}{(p^2 + E^*)^2} \geq \int_{B_{E^*}} \frac{d^3p}{(p^2 + E^*)^2} = C'(E^*)^{-1/2},$$

where  $B_{E^*}$  denotes the ball of radius  $\sqrt{E^*}$  around the origin ( $E^*$  is assumed to be small). Therefore we obtain

$$E \leq \lambda^2 \int_{\mathbb{T}^3} \frac{d^3p}{e(p)} + E^* - C'\lambda^2 \int_0^{E^*} t^{-1/2}dt,$$

so that if

$$E \geq \lambda^2 \int_{\mathbb{T}^3} \frac{d^3p}{e(p)} + \lambda^{4-\epsilon},$$

then  $E^* \geq \tilde{C}\lambda^{4-\epsilon}$  for small enough values of  $\lambda$ , as follows from the solution of the corresponding quadratic equation for  $\sqrt{E^*}$ .

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